



Delay-dependent Robust Stabilization for Uncertain T-S Fuzzy Systems with Additive Time Varying Delays

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Abstract: This paper investigates the problem of robust stabilization for uncertain Takagi–Sugeno (T–S) fuzzy systems with additive time varying delays. An appropriate Lyapunov-Krasovskii function is considered for solving this problem, and obtains considerably less conservative results than existing methods. The proposed approach constructs a new Lyapunov-Krasovskii functional using two additive delay components, and no free weighting matrices are employed in the theoretical result derivation. This reduces the number of scalar decision variables in linear matrix inequalities. The fuzzy state feedback gain is derived through the numerical solution of a set of linear matrix inequalities (LMIs). Finally, numerical examples are provided to illustrate the effectiveness of the proposed method, and to allow comparison with previous works.

Keywords: additive time varying delays, T-S fuzzy control systems, liner matrix inequality (LMI), stabilization

Introduction

Over the past few decades, considerable attention has focused on fuzzy systems in the form of the Takagi–Sugeno (T–S) model [1]. The TS model-based control method has been shown to provide an effective way to represent some complex non-linear systems. It combines flexible fuzzy logic theory and fruitful linear system theory into a unified framework to approximate complex nonlinear systems, especially those with incomplete information, and thus has emerged as an effective way to model and control complex systems, including time delay systems. The stability issue of T–S fuzzy control systems in non-linear stability frameworks has been studied extensively, e.g. [2–4].

Time delay can introduce instability in dynamic systems and degrade system performance. This complex phenomenon appears frequently in many industrial and engineering systems, such as aircraft stabilization, manual controls, laser models, neural networks, nuclear reactors, rolling mill systems, and communication networks [5, 6]. Recently, stability analysis and stabilizing controller design for T-S fuzzy systems with time-varying delay have

attracted considerable attention for use in time delays systems ([7–12]). Some proposed approaches provide stability analysis and controller synthesis of T–S fuzzy systems with time-varying delays [13–16]. For instance, in [17], the authors applied the Lyapunov–Razumikhin functional approach to stability analysis and stabilization. Tian and Peng [18] used the Lyapunov–Krasovskii functional method to investigate delay-dependent stability and controller design problems in uncertain nonlinear time-varying delay systems via T–S fuzzy models. In addition, much effort has been recently devoted to the development of the free-weighting matrix method. Efforts in [7,8,18,19] theoretically applied free-weighting matrixes to investigate stability and stabilization for systems with interval time-varying delays. The methods presented in [7,8,18,19] have been shown to be more effective than the previous method.

On the other hand, the TS fuzzy systems with time-varying delays have been modelled as systems with a single delay term in the state vector. More recently, the authors in [20] modeled the TS fuzzy system with two additive delay components, which have developed sufficient conditions for asymptotic stability and robust



stability analysis by using the Lyapunov-Krasovskii functional method and Finsler’s lemma. This approach produced a more effective less conservative result than was obtained in some existing methods.

This paper deals with delay-dependent stability and controller design problems for uncertain T–S fuzzy systems with two additional delay components. Sufficient stability analysis and controller design criteria are derived based on the Lyapunov–Krasovskii functional approach and Jensen’s inequality. By solving a set of LMIs, the state feedback gain and the upper bound of the time delay can be obtained. We provide two illustrative examples to show that the proposed design method is less conservative.

Lemma 1 ([6], Jensen’s inequality): for any constant matrix $M = M^T \in R^{n \times n}$, $M > 0$, scalar $\eta > 0$, vector function $\omega: [0, \eta] \rightarrow R^n$ such that the integrations in the following are well defined, then :

$$\eta \int_0^\eta \omega^T(\beta) M \omega(\beta) d\beta \geq \left[\int_0^\eta \omega(\beta) d\beta \right]^T M \left[\int_0^\eta \omega(\beta) d\beta \right]$$

Lemma 2 [21]: Let $Q = Q^T$, H , E , and $F(t)$ satisfying $F^T(t)F(t) \leq I$ are appropriately dimensioned matrices, with the following inequality :

$$Q + HF(t)E + E^T F^T(t)H^T < 0$$

is true, if and only if the following inequality holds for any scalar $\varepsilon > 0$

$$Q + \varepsilon^{-1} H H^T + \varepsilon E^T E < 0$$

System description and preliminaries

Consider the uncertain T–S fuzzy model, composed of a set of fuzzy implications, where each implication is expressed by a linear system model. The plant rule l of the

T–S fuzzy model is described by the following IF – THEN statement:

Plant Rule i :

IF $z_1(t)$ is W_1^i and ... and $z_g(t)$ is W_g^i THEN

$$\begin{cases} \dot{x}(t) = (A_{0i} + \Delta A_{0i}(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - h_1(t) - h_2(t)) \\ \quad + (B_i + \Delta B_i(t))u(t) \\ x(t) = \phi(t), t \in [-\bar{h}, 0], \quad i = 1, 2, \dots, r \end{cases} \quad (1)$$

where $z_1(t), z_2(t), \dots, z_g(t)$ are the premise variables, $W_j^i, j = 1, 2, \dots, g$ are fuzzy sets, $x(t) \in R^n$ is the state variable, and $u(t) \in R^m$ is the control input. $A_{0i}, A_{di} \in R^{n \times n}, B_i \in R^{n \times m}$, r is the number of if-then rules, $\phi(t)$ is a vector-valued initial condition of systems (1), and $h_1(t)$ and $h_2(t)$ are the time-varying delays satisfying

$$\begin{aligned} 0 \leq h_1(t) \leq \bar{h}_1, \quad \dot{h}_1(t) \leq d_1, \quad 0 \leq h_2(t) \leq \bar{h}_2, \quad \dot{h}_2(t) \leq d_2, \\ \bar{h} = \bar{h}_1 + \bar{h}_2 \quad \text{and} \quad d = d_1 + d_2 \end{aligned} \quad (2)$$

The parametric uncertainties $\Delta A_{0i}(t)$, $\Delta A_{di}(t)$ and $\Delta B_i(t)$ are time-varying matrices with appropriate dimensions, which can be described as :

$$\begin{bmatrix} \Delta A_{0i}(t) & \Delta A_{di}(t) & \Delta B_i(t) \end{bmatrix} = D_i F_i(t) \begin{bmatrix} E_{0i} & E_{di} & E_{bi} \end{bmatrix}, \quad i = 1, 2, \dots, r \quad (3)$$

where D_i, E_{0i}, E_{di} and E_{bi} are known constant real matrices with appropriate dimensions and $F_i(t)$ are unknown real time-varying matrices with Lebesgue measurable elements bounded by:

$$F_i^T(t)F_i(t) \leq I, \quad i = 1, 2, \dots, r \quad (4)$$

For simplicity, we introduce the following notations:

$$\bar{A}_{0i} = A_{0i} + \Delta A_{0i}(t), \quad \bar{A}_{di} = A_{di} + \Delta A_{di}(t)$$

$$\text{and } \bar{B}_i = B_i + \Delta B_i(t)$$

By using the center-average defuzzifier, product inference and singleton fuzzifier, the global dynamics of T–Z fuzzy system (1) can be expressed as

$$\dot{x}(t) = \sum_{i=1}^r \mu_i(z(t)) [\bar{A}_{0i} x(t) + \bar{A}_{di} x(t - h_1(t) - h_2(t)) + \bar{B}_i u(t)] \quad (5)$$

where, $\mu_i(z(t)) = \omega_i(z(t)) / \sum_{i=1}^r \omega_i(z(t))$, $\omega_i(z(t)) = \prod_{j=1}^g W_j^i(z_j(t))$, $z(t) = [z_1(t), \dots, z_g(t)]^T$ and $W_j^i(z_j(t))$ is the membership value of $z_j(t)$ in W_j^i . Some basic properties of $\mu_i(z(t))$ are $\mu_i(z(t)) \geq 0$, $\sum_{i=1}^r \mu_i(z(t)) = 1$. The i th controller rule is:

Control Rule i :

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IF $z_1(t)$ is W_1^i and ... and $z_g(t)$ is W_g^i

Then $u(t) = K_i x(t), \quad i = 1, 2, \dots, r,$ (6)

The defuzzified output of controller rule (6) is proposed as

$$u(t) = \sum_{j=1}^r \mu_j(z(t)) K_j x(t) \tag{7}$$

where K_j ($j = 1, 2, \dots, r$) are the local controller gains to be determined. The fuzzy controller is designed to determine the feedback gains K_j ($j = 1, 2, \dots, r$) such that the closed-loop system is asymptotically stable.

Combining (5) and (7), we obtain the following closed-loop fuzzy system:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i(z(t)) \mu_j(z(t)) [(\bar{A}_{0i} + \bar{B}_i K_j) x(t) + \bar{A}_{di} x(t - h_1(t) - h_2(t))] \\ x(t) = \phi(t), \quad t \in [-\bar{h}, 0] \quad i = 1, 2, \dots, r \end{cases} \tag{8}$$

Remark 1: The time delay verifying (2) represents the real situation in many practical applications. For example, in networked controlled systems successive delays with different properties are introduced into signal transmission, either from sensor to controller $h_1(t)$ or from controller to actuator $h_2(t)$, and these delays are actually time-varying. The stability analysis of such system has intervened out by adding up all the successive delays, i.e., $h(t) = h_1(t) + h_2(t)$. However, to the best of our knowledge, few papers have considered the additive time-varying delay case for T-S fuzzy systems.

Robust stability analysis and synthesis

This section analyzes the stability of closed-loop T-S fuzzy time-varying systems with two additive time varying delays (8), based on a new Lyapunov-Krasovskii functional approach.

Theorem 1. For given scalars $\bar{h}_1 \geq 0, \bar{h}_2 \geq 0$ and matrices K_j , system (8) satisfying conditions (2) is robustly asymptotically stable, if there exist symmetric positive definite matrices P, Q_1, Q_2, Z_1 and Z_2 with appropriate dimensions, and scalar $\varepsilon_{ij} > 0$ such that $Q_1 - Q_2 \geq 0$ and the following linear matrix inequalities (LMIs) hold for $i, j = 1, 2, \dots, r,$

$$\Sigma_{ij} = \begin{bmatrix} \Sigma_{11}^{ij} & \Sigma_{12}^{ij} & \Sigma_{13}^{ij} & \Sigma_{14}^{ij} & \Sigma_{15}^{ij} & \Sigma_{16}^{ij} \\ * & \Sigma_{22}^{ij} & \Sigma_{23}^{ij} & 0 & 0 & 0 \\ * & * & \Sigma_{33}^{ij} & \Sigma_{34}^{ij} & 0 & \Sigma_{36}^{ij} \\ * & * & * & \Sigma_{44}^{ij} & \Sigma_{45}^{ij} & 0 \\ * & * & * & * & \Sigma_{55}^{ij} & 0 \\ * & * & * & * & * & \Sigma_{66}^{ij} \end{bmatrix} < 0 \tag{9}$$

where,

$$\begin{aligned} \Sigma_{11}^{ij} &= P(A_{0i} + B_i k_j) + (A_{0i} + B_i k_j)^T P - Z_1 + Q_1, \\ \Sigma_{12}^{ij} &= Z_1, \quad \Sigma_{13}^{ij} = P A_{di}, \quad \Sigma_{14}^{ij} = (A_{0i} + B_i K_j)^T P, \\ \Sigma_{15}^{ij} &= P D_i, \quad \Sigma_{16}^{ij} = \varepsilon_{ij} (E_{0i} + E_{bi} K_j)^T \\ \Sigma_{22}^{ij} &= -Z_1 - Z_2 - (1 - d_1)(Q_1 - Q_2), \quad \Sigma_{23}^{ij} = Z_2 \\ \Sigma_{33}^{ij} &= -Z_2 - (1 - d_1 - d_2) Q_2, \quad \Sigma_{34}^{ij} = A_{di}^T P, \quad \Sigma_{36}^{ij} = \varepsilon_{ij} E_{di}^T \\ \Sigma_{44}^{ij} &= h_1^2 Z_1 + h_2^2 Z_2 - 2P, \quad \Sigma_{45}^{ij} = P D_i \\ \Sigma_{55}^{ij} &= -\varepsilon_{ij} I, \quad \Sigma_{66}^{ij} = -\varepsilon_{ij} I \end{aligned}$$

Proof: Define the following Lyapunov-Krasovskii functional

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{10}$$

where,

$$V_1(x_t) = x^T(t) P x(t) \tag{11}$$

$$\begin{aligned} V_2(x_t) &= \bar{h}_1 \int_{-\bar{h}_1}^0 \int_{t+\theta}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds d\theta \\ &+ \bar{h}_2 \int_{-\bar{h}_1 - \bar{h}_2}^{-\bar{h}_1} \int_{t+\theta}^t \dot{x}^T(s) Z_2 \dot{x}(s) ds d\theta \end{aligned} \tag{12}$$

$$V_3(x_t) = \int_{t-h_1(t)}^t x^T(s) Q_1 x(s) ds + \int_{t-h_1(t)-h_2(t)}^{t-h_1(t)} x^T(s) Q_2 x(s) ds \tag{13}$$

computes the time derivative of $V(x_t)$ along the trajectory of Eq. (8) to obtain,

$$\dot{V}_1(x_t) = 2x^T(t) P \times \left\{ \sum_{i=1}^r \sum_{j=1}^r \mu_i(z(t)) \mu_j(z(t)) [(\bar{A}_{0i} + \bar{B}_i K_j) x(t) + \bar{A}_{di} x(t - h_1(t) - h_2(t))] \right\} \tag{14}$$

$$\begin{aligned} \dot{V}_2(x_t) &= \bar{h}_1 \int_{-\bar{h}_1}^0 \left[\dot{x}^T(t) Z_1 \dot{x}(t) - \dot{x}^T(t + \theta) Z_1 \dot{x}(t + \theta) \right] d\theta + \\ &\bar{h}_2 \int_{-\bar{h}_1 - \bar{h}_2}^{-\bar{h}_1} \left[\dot{x}^T(t) Z_2 \dot{x}(t) - \dot{x}^T(t + \theta) Z_2 \dot{x}(t + \theta) \right] d\theta \end{aligned}$$

$$\begin{aligned} \dot{V}_3(x_t) &= \bar{h}_1^2 \dot{x}^T(t) Z_1 \dot{x}(t) - \bar{h}_1 \int_{t-\bar{h}_1}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds + \\ &\bar{h}_2^2 \dot{x}^T(t) Z_2 \dot{x}(t) - \bar{h}_2 \int_{t-\bar{h}_1}^{t-\bar{h}_2} \dot{x}^T(s) Z_2 \dot{x}(s) ds \end{aligned} \tag{15}$$

Z_1 and Z_2 are symmetric positive definite matrices, and the following inequalities always hold [22]:

$$- \int_{t-\bar{h}_1}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds \leq - \int_{t-h_1(t)}^t \dot{x}^T(s) Z_1 \dot{x}(s) ds$$



$$-\int_{t-\bar{h}}^{t-\bar{h}_1} \dot{x}^T(s)Z_2\dot{x}(s)ds \leq -\int_{t-h(t)}^{t-h_1(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds$$

where $h(t) = h_1(t) + h_2(t)$. Using the above inequalities in (15) we have:

$$\begin{aligned} \dot{V}_2(x_t) \leq & \bar{h}_1^2 \dot{x}^T(t)Z_1\dot{x}(t) - \bar{h}_1 \int_{t-h_1(t)}^t \dot{x}^T(s)Z_1\dot{x}(s)ds + \\ & \bar{h}_2^2 \dot{x}^T(t)Z_2\dot{x}(t) - \bar{h}_2 \int_{t-h(t)}^{t-h_1(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \end{aligned} \quad (16)$$

By using lemma 1 we obtain:

$$\begin{aligned} \dot{V}_2(x_t) \leq & \bar{h}_1^2 \dot{x}^T(t)Z_1\dot{x}(t) - [x(t) - x(t-h_1(t))]^T Z_1 [x(t) - x(t-h_1(t))] + \\ & \bar{h}_2^2 \dot{x}^T(t)Z_2\dot{x}(t) - [x(t-h_1(t)) - x(t-h(t))]^T Z_2 [x(t-h_1(t)) - x(t-h(t))] \end{aligned} \quad (17)$$

$$\begin{aligned} \dot{V}_3(x_t) = & x^T(t)Q_1x(t) - (1-\dot{h}_1(t))x^T(t-h_1(t))Q_1x(t-h_1(t)) \\ & + (1-\dot{h}_1(t))x^T(t-h_1(t))Q_2x(t-h_1(t)) - (1-\dot{h}_1(t)-\dot{h}_2(t)) \\ & x^T(t-h(t))Q_2x(t-h(t)) \\ \dot{V}_3(x_t) \leq & x^T(t)Q_1x(t) - (1-d_1)x^T(t-h_1(t))(Q_1-Q_2)x(t-h_1(t)) \\ & - (1-d_1-d_2)x^T(t-h(t))Q_2x(t-h(t)) \end{aligned} \quad (18)$$

where $Q_1 - Q_2 \geq 0$

Now let,

$$\xi^T(t) = [x^T(t) \quad x^T(t-h_1(t)) \quad x^T(t-h(t)) \quad \dot{x}^T(t)]$$

Taking account of (14), (17) and (18), along with the following result

$$-2\dot{x}^T(t)P\dot{x}(t) + 2\dot{x}^T(t)P \times \left\{ \begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r \mu_i(z(t))\mu_j(z(t))[(\bar{A}_{0i} + \bar{B}_iK_j)x(t) + \\ & \bar{A}_{di}x(t-h_1(t)-h_2(t))] \end{aligned} \right\} = 0$$

We thus obtain

$$\dot{V}(x_t) \leq \sum_{i=1}^r \sum_{j=1}^r \mu_i(z(t))\mu_j(z(t))\xi^T(t)\Pi_{ij}\xi(t) \quad (19)$$

where

$$\Pi_{ij} = \begin{bmatrix} \Pi_{11}^{ij} & \Sigma_{12}^{ij} & \Pi_{13}^{ij} & \Pi_{14}^{ij} \\ * & \Sigma_{22}^{ij} & \Sigma_{23}^{ij} & 0 \\ * & * & \Sigma_{33}^{ij} & \Pi_{34}^{ij} \\ * & * & * & \Sigma_{44}^{ij} \end{bmatrix}$$

$$\begin{aligned} \Pi_{11}^{ij} = & P(\bar{A}_{0i} + \bar{B}_iK_j) + (\bar{A}_{0i} + \bar{B}_iK_j)^T P - Z_1 + Q_1, \\ \Pi_{13}^{ij} = & P\bar{A}_{di}, \quad \Pi_{14}^{ij} = (\bar{A}_{0i} + \bar{B}_iK_j)^T P, \quad \Pi_{34}^{ij} = \bar{A}_{di}^T P \text{ and} \\ \Sigma_{12}^{ij}, & \Sigma_{22}^{ij}, \Sigma_{23}^{ij}, \Sigma_{33}^{ij}, \Sigma_{44}^{ij} \text{ are defined in (9).} \end{aligned}$$

Decomposing the resulting matrix Π_{ij} into nominal and uncertain parts leads to

$$\Pi_{ij} = \Xi_{ij} + H_i F_i(t) E_{ij} + E_{ij}^T F_i^T(t) H_i^T \quad (20)$$

where,

$$\begin{aligned} \Xi_{ij} = & \begin{bmatrix} \Sigma_{11}^{ij} & \Sigma_{12}^{ij} & \Sigma_{13}^{ij} & \Sigma_{14}^{ij} \\ * & \Sigma_{22}^{ij} & \Sigma_{23}^{ij} & 0 \\ * & * & \Sigma_{33}^{ij} & \Sigma_{34}^{ij} \\ * & * & * & \Sigma_{44}^{ij} \end{bmatrix}, \\ H = & [D_i^T P \quad 0 \quad 0 \quad D_i^T P]^T, \\ E_{ij} = & [(E_{0i} + E_{bi}) \quad 0 \quad E_{di} \quad 0] \end{aligned}$$

Applying Lemma 2, it is clear that the following inequalities always hold

$$\Pi_{ij} = \Xi_{ij} + H_i F_i(t) E_{ij} + E_{ij}^T F_i^T(t) H_i^T < \Xi_{ij} + H_i \varepsilon_{ij}^{-1} H_i^T + E_{ij}^T \varepsilon_{ij} E_{ij} \quad (21)$$

Using Schur's complement of (21), LMI (9) is equivalent to Π_{ij} in (19), then $\dot{V}(x_t) < 0$. Thus, system (8) is robustly asymptotically stable and the proof is complete.

Remark 2: The authors in [20] proposed delay-dependent stability criteria for T S fuzzy systems with additive time-varying delay, in which free weighting matrices are introduced and less conservative results are obtained as compared with some existing results [16,18,23]. The approach developed in the present paper is based on the Lyapunov Krasovskii functional method without any free weighting matrices. Theorem 1 of this paper and theorem 1 of [20] respectively require $\frac{5n(n+1)}{2}$ and

$3n^2 + \frac{5n(n+1)}{2}$ variables. Thus, theorem 1 requires fewer

variables than $3n^2$. Therefore, our results alleviate the computational demand required to obtain a solution for stable conditions. This advantage is especially apparent in systems with a large dimension n. Remark 3: In lemma 1, the proposed approach accounts for useful terms in the derivative of Lyapunov-Krasovskii functional, e.g.

$$\int_{t-\bar{h}_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds \text{ and } \int_{t-\bar{h}}^{t-\bar{h}_1} \dot{x}^T(s)Z_2\dot{x}(s)ds \text{ in (15). Some}$$

previous studies [17,24,25] estimate the derivative of $\int_{-\bar{h}_1}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta$ as

$$\dot{x}^T(t)Z_1\dot{x}(t) - \bar{h}_1 \int_{t-\bar{h}_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds \text{ and the term}$$

$$-\int_{t-\bar{h}_1}^t \dot{x}^T(s)Z_1\dot{x}(s)ds \text{ is neglected, which may lead to}$$

considerable conservativeness. As shown in the examples, our proposed approach is expected to reduce conservatism.



Robust controller design

This section develops the problem of stabilization for closed-loop T–S fuzzy systems, such that the obtained controller will guarantee the asymptotical stability of the closed-loop system (8).

Theorem 2: For given scalars $\bar{h}_1 \geq 0, \bar{h}_2 \geq 0$, the closed-loop systems (8) satisfying condition (2) is robustly asymptotically stable with feedback gains $K_j = Y_j X^{-T}$ ($j = 1, 2, \dots, r$), if there exist symmetric positive definite matrices $X, \tilde{Q}_1, \tilde{Q}_2, \tilde{Z}_1, \tilde{Z}_2$ with appropriate dimensions, and scalar $\varepsilon_{ij} > 0$ such that $\tilde{Q}_1 - \tilde{Q}_2 \geq 0$ and the following linear matrix inequalities (LMIs) hold for $i, j = 1, 2, \dots, r$.

$$\tilde{\Sigma}_{ij} = \begin{bmatrix} \tilde{\Sigma}_{11}^{ij} & \tilde{\Sigma}_{12}^{ij} & \tilde{\Sigma}_{13}^{ij} & \tilde{\Sigma}_{14}^{ij} & \tilde{\Sigma}_{15}^{ij} & \tilde{\Sigma}_{16}^{ij} \\ * & \tilde{\Sigma}_{22}^{ij} & \tilde{\Sigma}_{23}^{ij} & 0 & 0 & 0 \\ * & * & \tilde{\Sigma}_{33}^{ij} & \tilde{\Sigma}_{34}^{ij} & 0 & \tilde{\Sigma}_{36}^{ij} \\ * & * & * & \tilde{\Sigma}_{44}^{ij} & \tilde{\Sigma}_{45}^{ij} & 0 \\ * & * & * & * & \tilde{\Sigma}_{55}^{ij} & 0 \\ * & * & * & * & * & \tilde{\Sigma}_{55}^{ij} \end{bmatrix} < 0 \quad (22)$$

where,

$$\begin{aligned} \tilde{\Sigma}_{11}^{ij} &= A_{0i} X + X A_{0i}^T + B_i Y_j + Y_j^T B_i^T - \tilde{Z}_1 + \tilde{Q}_1, \\ \tilde{\Sigma}_{12}^{ij} &= \tilde{Z}_1, \quad \tilde{\Sigma}_{13}^{ij} = A_{di} X^T, \quad \tilde{\Sigma}_{14}^{ij} = X A_{0i}^T + Y_j^T B_i, \\ \tilde{\Sigma}_{15}^{ij} &= D_i, \quad \tilde{\Sigma}_{16}^{ij} = \varepsilon_{ij} (X E_{0i}^T + Y_j^T E_{bi}^T), \\ \tilde{\Sigma}_{22}^{ij} &= -\tilde{Z}_1 - \tilde{Z}_2 - (1 - d_1)(\tilde{Q}_1 - \tilde{Q}_2), \quad \tilde{\Sigma}_{23}^{ij} = \tilde{Z}_2 \\ \tilde{\Sigma}_{33}^{ij} &= -\tilde{Z}_2 - (1 - d_1 - d_2)\tilde{Q}_2, \quad \tilde{\Sigma}_{34}^{ij} = X A_{di}^T, \\ \tilde{\Sigma}_{35}^{ij} &= \varepsilon_{ij} X E_{di}^T, \quad \tilde{\Sigma}_{44}^{ij} = h_1^2 \tilde{Z}_1 + h_2^2 \tilde{Z}_2 - 2X^T, \quad \tilde{\Sigma}_{45}^{ij} = D_i, \\ \tilde{\Sigma}_{55}^{ij} &= -\varepsilon_{ij} I, \quad \tilde{\Sigma}_{66}^{ij} = -\varepsilon_{ij} I \end{aligned}$$

Proof: Pre- and post-multiply both sides of Eq. (9) with $\text{diag}[X \ X \ X \ X \ I \ I]$ and its transpose. Letting $X = P^{-1}, X Z_1 X^T = \tilde{Z}_1, X Z_2 X^T = \tilde{Z}_2, X Q_1 X^T = \tilde{Q}_1, X Q_2 X^T = \tilde{Q}_2$ and $Y_j = K_j X^T$ we arrive at (22) and the proof is complete.

Numerical examples

This section gives two numerical examples to demonstrate the effectiveness of the proposed approach by theorem 1 and theorem 2.

Example 1: Consider a system with the following rules: [16, Example 1]

Rule 1: If $x_1(t)$ is W_1

Then $\dot{x}(t) = A_{01}x(t) + A_{d1}x(t - h_1(t) - h_2(t))$

Rule 2: If $x_2(t)$ is W_2

Then $\dot{x}(t) = A_{02}x(t) + A_{d2}x(t - h_1(t) - h_2(t))$

and the membership functions for rule 1 and rule 2 are

$$\mu_1(z(t)) = \frac{1}{1 + \exp(-2x_1(t))}, \quad \mu_2(z(t)) = 1 - \mu_1(z(t))$$

where,

$$A_{01} = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$A_{02} = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix},$$

$$A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}$$

Our purpose is to calculate the upper bound \bar{h}_1 of $h_1(t)$, or \bar{h}_2 of $h_2(t)$, when the other is known. The maximum allowable upper bound of the system is obtained by adding \bar{h}_1 and \bar{h}_2 , i.e., $\bar{h} = \bar{h}_1 + \bar{h}_2$.

Table 1 lists the results of the maximum allowable delay bounds $\bar{h} = \bar{h}_1 + \bar{h}_2$ derived from various methods including, Tian and Peng [18], Peng et al. [16], Fan et al. [23], Idrissi and Tissir [20] and the one proposed in this paper. Table 1 shows that the results obtained from our method are less conservative than those obtained from existing methods. Figure 1 shows the response of the fuzzy system with $\bar{h} = 1.897$ and initial condition $x(t) = [1 \ 0.2]^T, t \in [-1.987 \ 0]$.

Now, let us consider the number of variables in different methods. Generally speaking, for the same upper delay bound \bar{h} , fewer variables require less computational power. While Idrissi and Tissir [20] required 27 variables in Theorem 1, our proposed approach requires only 15, and provides a better upper delay bound \bar{h} with fewer variables as compared to 23 for Peng et al. [16] (Corollary 1) and 38 for Fan et al. [23].

The simulation result shows that system (Example 1) is asymptotically stable for the upper bound $\bar{h} = 1.897$, see Fig. 1.

Example 2: Consider the following T–S fuzzy model with additive time delay [17, Example 3]

Rule 1: If $(x_2(t)/0.5)$ is about 0.5

Then

$$\dot{x}(t) = (A_{01} + \Delta A_{01})x(t) + (A_{d1} + \Delta A_{d1})x(t - h_1(t) - h_2(t)) + B_1 u(t)$$

Rule 2: If $(x_2(t)/0.5)$ is about π or $-\pi$

Then

$$\dot{x}(t) = (A_{02} + \Delta A_{02})x(t) + (A_{d2} + \Delta A_{d2})x(t - h_1(t) - h_2(t)) + B_2 u(t)$$

where,

$$A_{01} = \begin{bmatrix} -0 & 1 \\ 0.1 & -2 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.5 - 1.5\beta \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_{d1} = A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \quad \beta = \frac{0.01}{\pi},$$

$$D_i = \begin{bmatrix} -0.03 & 0 \\ 0 & 0.03 \end{bmatrix}, E_{0i} = \begin{bmatrix} -0.15 & 0.2 \\ 0 & 0.04 \end{bmatrix},$$

$$E_{di} = \begin{bmatrix} -0.05 & -0.35 \\ 0.08 & -0.45 \end{bmatrix}, i = 1, 2.$$

and β is used to prevent system matrices from being singular.

The membership functions are set as follows:

$$\mu_1(z(t)) = \left(1 - \frac{1}{1 + \exp(-3((x_2(t)/0.5) - 0.5\pi))} \right)$$

$$\times \left(\frac{1}{1 + \exp(-3((x_2(t)/0.5) + 0.5\pi))} \right)$$

$$\mu_2(z(t)) = 1 - \mu_1(z(t))$$

when $d = 0$, (e.g. $d_1 = 0$ and $d_2 = 0$). Table 2 lists the results using Theorem 2 of this paper, where $\Delta A_{0i} = \Delta A_{di} = 0$, and the obtained results improve on those from previous works [17,19,24,25].

Table 3 lists the results from using Theorem 2 of this paper, when $\Delta A_{0i} \neq 0$ and $\Delta A_{di} \neq 0$.

We can see from Tables 2 and 3 that our proposed method provides less conservative results and smaller feedback gains than those derived from other previous methods [17,19,24,25].

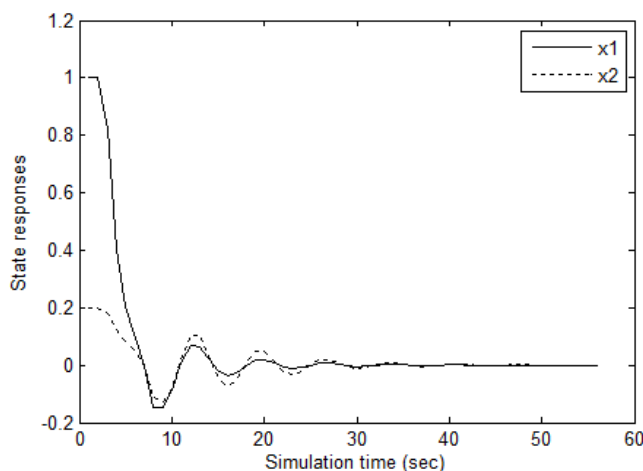


Figure 1. System time response (Example 1).

Table 1. Calculated upper bound \bar{h} for different cases.

Methods	Upper bound \bar{h}						Num.var
	upper bound \bar{h}_2 for given \bar{h}_1			upper bound \bar{h}_1 for given \bar{h}_2			
Tian and Peng [18](Corollary 1)	1.597						60
Peng et al. [16] (Corollary 1)	1.597						23
Fan et al. [23]	1.597						38
-	upper bound \bar{h}_2 for given \bar{h}_1			upper bound \bar{h}_1 for given \bar{h}_2			-
Idrissi and Tissir [20]	$\bar{h}_1 = 1$	$\bar{h}_1 = 1,2$	$\bar{h}_1 = 1,5$	$\bar{h}_2 = 0,2$	$\bar{h}_2 = 0,3$	$\bar{h}_2 = 0,5$	-
	0,897	0,667	0,208	1,504	1,449	1,323	27
Theorem 1 of this paper	0,897	0,667	0,208	1,504	1,449	1,323	15

Table 2. Calculated upper bound \bar{h} and controller feedback gains without uncertainties (example 2).

Methods	Upper bound \bar{h}	K_1	K_2
Lin et al [17]	1.725	[30.6571 10.0442]	[30.6589 11.1803]
Guan and Chen [24]	3.4385	[-0.9390 0.3123]	[-0.3277 -0.3636]
Chen et al [25]	25.7865	[-1.2141 0.8750]	[-1.2141 -0.6202]
Li et al [19]	25.7865	[-0.9318 0.1265]	[-0.9318 -1.3687]
	26.8617	[-0.9211 0.1344]	[-0.9211 -1.3609]
-	Upper bound $\bar{h} = \bar{h}_1 + \bar{h}_2$	-	-
Theorem 2 of this paper	$\bar{h}_1 = 20, \bar{h}_2 = 6.8617$	[-0.7328 -0.9645]	[-0.8061 -1.0609]
	$\bar{h}_1 = 7.1254, \bar{h}_2 = 22$	[-0.7269 -0.8968]	[-0.7996 -0.9865]



Table 3. Calculated upper bound \bar{h} and controller feedback gains with uncertainties (Example 2).

Methods	Upper bound \bar{h}	K_1	K_2
Chen et al [25]	20.5640	$[-1.3987 \quad -0.6601]$	$[-1.3991 \quad -2.1607]$
Li et al [19]	20.5640	$[-1.3778 \quad -1.9868]$	$[-1.3778 \quad -3.4820]$
	22.4582	$[-1.4470 \quad -2.0154]$	$[-1.4470 \quad -3.5106]$
-	Upper bound $\bar{h} = \bar{h}_1 + \bar{h}_2$	-	-
Theorem 2 of this paper	$\bar{h}_1 = 18, \bar{h}_2 = 4.4582$	$[-0.9449 \quad -1.2548]$	$[-0.9348 \quad -1.2755]$
	$\bar{h}_1 = 13, \bar{h}_2 = 15.4263$	$[-0.9298 \quad -1.1811]$	$[-0.9185 \quad -1.2058]$

Conclusion

A delay-dependent stability analysis and is presented to resolve controller design problems for uncertain T-S fuzzy systems with two additive time varying delays. The proposed LMIs are obtained using the Lyapunov Krasovskii functional method and improved Jensen's inequality. Moreover, the maximum allowable upper delay bound and the feedback controller gain can be simultaneously obtained by solving the LMI set. The reduced conservativeness of the results is shown by two numerical examples.

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